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Abstract: A kinetic theory approach to collision processes in ionized and neutral gases is presented. This approach is adequate for the unified treatment of the dynamic properties of gases over a continuous range of pressures from the Knudsen limit to the high-pressure limit where the aerodynamic equations are valid. It is also possible to satisfy the correct microscopic boundary conditions. The method consists in altering the collision terms in the Boltzmann equation. The modified collision terms are constructed so that each collision conserves particle number, momentum and energy; other characteristics such as persistence of velocities and angular dependence may be included. The present article illustrates the technique for a simple model involving the assumption of a collision time independent of velocity; this model is applied to the study of small amplitude oscillations of one-component ionized and neutral gases. The initial value problem for unbounded space is solved by performing a Fourier transformation on the space variables and a Laplace transformation on the time variable. For uncharged gases there results the correct adiabatic limiting law for sound-wave propagation at high pressures and, in addition, one obtains a theory of absorption and dispersion of sound for arbitrary pressures. For ionized gases

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the difference in the nature of the organization in the low-pressure plasma oscillations and in high-pressure sound-type oscillations is studied. Two important cases are distinguished. If the wavelength of the oscillations is long compared to either the Debye length or the mean free path, a small change in frequency is obtained as the collision frequency varies from zero to infinity. The accompanying absorption is small; it reaches its maximum value when the collision frequency equals the plasma frequency. The second case refers to waves shorter than both the Debye length and the mean free path; these waves are characterized by a very heavy absorption.

INTRODUCTION

The dynamical theory of gases may be studied from two points of view. One may take as starting point the macroscopic equations of aerodynamics with the density ρ , mass velocity $\vec{q}$ and temperature T as independent variables, and involving various coefficients, e.g., viscosity, heat conduction, etc. On the other hand, one may use a more fundamental and general microscopic formalism. The most fruitful of such formalisms available at present is that in terms of one particle distribution functions satisfying integro-differential equations of the Boltzmann type.

The aerodynamic equations are adequate for treating a certain wide range of problems in gas dynamics and most of the developments of the theory have been in this domain. In this range the microscopic theory would not yield significantly different results. In fact, subject to certain conditions which delimit their range of applicability, the aerodynamic equations, together with explicit formulae for the various coefficients entering into them, are derivable from the microscopic theory. A detailed account of this derivation is presented

in the book of Chapman and Cowling.[1]

There are, however many important situations in which the macroscopic theory does not give a correct description. During recent years interest in such problems has increased considerably. An extreme example of the breakdown of the aerodynamic theory, viz., in the Knudsen region, has been known for some time; here the mean free path of a molecule is large compared to some linear dimension of the apparatus and behavior at the boundaries becomes important. Other examples are provided by high-frequency sound waves in a rarefied gas and plasma oscillations.

The solution of the Boltzmann equation is, in general, a matter of considerable difficulty even in cases corresponding to the physically simplest situations. Significant progress has been confined practically to the study of two limiting cases in which two different approximation procedures can be applied. A criterion for the range of validity of the approximate methods is provided by the comparison of some characteristic time τ (or characteristic length L) with the average time τ_c (or mean free path L_c) between molecular collisions.

For high density ($\tau \gg \tau_c$ or L $\gg$ L_c) the Enskog-Chapman theory may be used. The first approximation of the theory consists in assuming local thermodynamic equilibrium and a common drift velocity for all molecular species. The next approximation corrects the distribution function by terms proportional to the first derivatives of temperature, velocity and density; this corresponds to the aerodynamic equations with coefficients of heat conduction, viscosity and diffusion. The high density region ($\tau \gg \tau_c$) is in fact the range in which the aerodynamic equations provide an adequate description. Higher approximations of the E-C theory lead to correction terms proportional to higher derivatives of

1) S. Chapman and T. G. Cowling, "The Mathematical Theory of Non-Uniform Gases", Cambridge University Press, London, 2nd Edition, 1952.

T, $\vec{q}$, ρ. The successive approximations of the E-C theory correspond to an expansion of the distribution function in powers of the mean free path L_c. If we consider sound waves with wavelength $L \gg L_c$, the first and second approximations are already sufficient to give all significant features of the process. However, when L becomes comparable with L_c, it is necessary to go to the third and even higher approximations to obtain adequate results; as discussed by Herzfeld[2] important dynamic contributions appear only in approximations higher than the second. The third approximation already involves formidable labor and has been used to solve only the simplest problems. Consideration of higher approximations is, in any case, of doubtful value as the entire procedure breaks down in just the range where the contributions from these higher-order terms become important. In addition, the boundary conditions in many problems cannot be specified properly within the scope of the E-C formalism. A different approach, using expansions in terms of Hermite polynomials in velocity space, has been given by Grad.[3] He uses some moments of low order in addition to the usual ones representing ρ, $\vec{q}$ and T. The procedure involves a gain in simplicity over the E-C theory but is still quite complicated. It is basically a high-density theory and is capable of dealing only with boundary conditions which can be specified in terms of the moments appearing in the theory. The limitations of any theory based on the use of a finite number of moments will be brought out in the discussions of this series of papers.

The opposite limiting case ($\tau \ll \tau_c$ or $L \ll L_c$) has been studied extensively by mean free-path methods. Jaffé[4] has shown that this case can be treated

2) K. F. Herzfeld, Ann. Physik 23, 465 (1935); cf. also H. Primakoff, J. Acoust. Soc. Am. 14, 14 (1942); H. Tsien and R. Schamberg, ibid. 18, 334 (1946); C. S. W. Chang and G. Uhlenbeck, Dept. Eng. Research, Univ. Michigan, APL/JHUCM-443, UMH-3-P, Feb., 1948; M. Kohler, Abh. Braunschweig. Wiss. Gesell. 2, 104 (1950).

3) H. Grad, Comm. Pure Applied Math. 2, 331 (1949).

4) H. Jaffé, Ann. Physik 6, 195 (1930).

from the point of view of the Boltzmann equation by expanding the distribution function in a series of inverse powers of the mean free path; the first approximation consists in neglecting collisions completely. In contrast to the high density limit, the problem can be solved subject to correct microscopic boundary conditions. The method becomes extremely complicated when one attempts to carry out higher approximations to give results which go much beyond those obtainable by simple mean free-path arguments. For this reason this method has not been applied to any great extent.

The low-density case is of particular interest for ionized gases where the coupling of the material motions with the electromagnetic field plays an essential and determining role. Ionized gases occur widely in nature and in the laboratory with densities and temperatures covering a large range. In laboratory discharge tubes and similar devices pressures ranging from 10^{-3} mm up to 10^4 mm are commonplace. In the negative glow of a discharge tube at 10^{-2} mm the material is at least singly ionized; the electron density is about 10^{11}/cc and the mean free path is of the order of a few cm. At pressures of the order of an atmosphere the ionization is only partial; the electron density is of the order of 10^{15}/cc and mean free paths are of the order of 10^{-5} cm. In nature the range covered is considerably wider.[5] The interstellar medium has a density varying from 10^{-24} gm /cc up to 10^{-21} gm/cc. The gas may be neutral as in the low-temperature HI regions, or completely ionized as in HII regions near very luminous stars. In the outer corona of the sun the degree of ionization is high and the electron density is of the order of 10^8/cc. As we proceed in towards the center of the star, the degree of ionization varies considerably. In the photo-spheric layers (mainly responsible for the sun's continuous spectrum), the particle density is of the order of 10^{11}/cc and the ionization is small. Near the center of the sun we of course have very much greater densities and temperatures

5) J. A. Hynek, "Astrophysics," McGraw-Hill Book Co., New York, 1951.

and a very high degree of ionization.

We thus see that it would be very desirable to have a theory capable of treating the whole range from low to high densities for mixtures of neutral and ionized gases. The treatment of an ionized gas involves certain new elements in addition to those for neutral gases; this will be discussed in detail in Sec. 2. The region of intermediate density ($\tau \simeq \tau_c$ or $L \simeq L_c$) has thus far been practically unexplored on account of excessive mathematical complications. The procedures discussed above for the two limiting cases of high and low density are not applicable in this region. For similar reasons, the solution of definite initial and boundary value problems with the Boltzmann equation has also not been studied to any extent. In order to make some progress in the investigation of such problems it is necessary to simplify the mathematical procedures quite considerably.

The aim of this series of papers is to propose a treatment of collision processes in gases which leads to just such a simple mathematical formalism. The main difficulty in handling the full Boltzmann equation arises from the complicated nature of the collision terms. In contrast to other procedures, ours consists in replacing this troublesome collision integral by mathematically simpler terms; these, however, are chosen so as to conform to the conservation laws for mass, momentum and energy and at the same time to represent certain essential features of collisions, e.g., persistence of velocity. This approach makes possible a survey of the whole range from low density to high density and including the intermediate region. It leads to the correct asymptotic behavior in the two limiting cases mentioned above. The mathematical simplification introduced with the model enables one to solve problems which are physically more complex than those soluble with the standard Boltzmann equation; in addition, one is able to treat definite initial and boundary-value problems.

Each collision term in a Boltzmann equation consists of two parts. The one part,[1] $-\iint f_1(\vec{v}_1, \vec{x}, t) \, f_2(\vec{v}_2, \vec{x}, t) \, k_{12} \, d\vec{k} \, d\vec{v}_2$, represents particles removed or absorbed from a definite velocity range by collisions; the other part represents the particles emitted into that range as a result of collisions. Our absorption term, $-n(\vec{x}, t) \, k(\vec{v}, \vec{x}, t) \, f(\vec{v}, \vec{x}, t)$, where $n(\vec{x}, t) = \int f(\vec{v}, \vec{x}, t) \, d\vec{v}$ is substantially the same as that in the Boltzmann equation. The emission term (which is the real source of difficulty) is replaced in our crudest model by a term representing a Maxwellian distribution of the emitted particles with a density, mass velocity and temperature satisfying mass, momentum and energy conservation requirements. This bears some resemblance to the "local thermo-dynamic equilibrium" model used in discussing the formation of the continuous spectrum in stellar atmospheres. The similarity of the Boltzmann equation to that governing propagation of radiation has been discussed by Jaffé.[4] The method proposed here is quite different, however, and is capable of considerable extension and generalization to include the study of physically complex problems. We shall show in a later paper that it may be used to provide a method of solution of the standard Boltzmann equation.

The plan of this series of papers is the following. In the first paper we study systems which are, or may be reduced to, one-component systems; to illustrate the technique we avoid mathematical complications by making the simplified assumption of constant collision time. This simple model is applied to the study of dispersion and absorption of sound in a simple unionized gas and also to the oscillations of an ionized gas with the additional simplifying assumption that the positive ions may be replaced by a uniform constant distribution of positive charge. The physically most significant feature here is the study of the transition from low-pressure plasma waves to high-pressure sound-type waves.

In the second paper we extend this treatment to a consideration of two- and three-component systems. This includes a discussion of the translational dispersion effect for sound waves in unionized gas mixtures. The further effect on small oscillations arising from the coupling between positive ions and neutrals will be studied over the whole range from low to high pressures; the two types of wave which occur at low pressures are found to merge into a single type of wave as the pressure increases and the collision mechanism begins to predominate.

Paper III will deal with the oscillations of ionized gases in static external magnetic fields. The range of validity of the conventional magnetohydrodynamic theory will be discussed there. An important point dealt with in that paper is the way in which gaps in the frequency spectrum, present at low densities, disappear as the density is increased.

The discussions of the first three papers are confined to initial value problems in the theory of small oscillations. The accent there is substantially on the study of the mechanism of co-operation in ionized and unionized gases. In Paper IV we abandon the assumption of constant collision time and examine physically more realistic models. The connection of our kinetic equations with the standard Boltzmann equation is discussed in more detail. In subsequent papers we shall study particular boundary-value problems and some nonlinear processes.

2. <u>EARLIER TREATMENTS OF SYSTEMS OF CHARGED PARTICLES</u>

The theory of oscillations in ionized gases is still in a rather tentative state. However, a number of features can be understood at least qualitatively. An essential difference from the theory for neutral gases is that approximately undamped waves can be propagated not only at high pressures $(\tau \gg \tau_c)$ but also at low pressures $(\tau \ll \tau_c)$.

The mechanisms responsible for the co-operative behavior of the medium are different in the extreme cases of low density and high density.[6] At low densities the collisions are only of secondary importance and the forces acting to change the state of motion of a particle are electromagnetic in character. Energy and momentum are transferred from one group of particles to another through the intermediary of the electromagnetic field in a way now familiar from the study of microwave devices. It is important to note, however, that the transfer of energy by the high-frequency components of a pulse is a very slow process in low-pressure plasmas. If one neglects the effect of random thermal motion, the group velocity of the waves is zero and a disturbance would remain completely localized. Including the effects of thermal motions, the group velocity of high-frequency waves is approximately $v_g = (3pa/\omega_p) a$, where $a = (kT/m)^{1/2}$; this is much smaller than the sound velocity, $5a/3$, for waves whose wave number p is smaller than the inverse of the Debye length (a/ω_p). In addition, waves with $p \simeq a/\omega_p$ are very heavily damped.

Let us now turn attention to attempts which have been made to treat the oscillations of ionized gases quantitatively by the methods of kinetic theory. It is clear that the long range of the Coulomb force implies that the concept of a definite binary collision will be inadequate except for very close encounters. Instead, every charged particle is continually interacting with all the particles of the assembly. An important theoretical step, making possible the approximate treatment of many properties of ionized gases, was taken by Vlasov.[7] Langmuir [8] had already pointed out that the Debye length is characteristic of the dynamic behavior as well as of the static properties of an ionized gas.

6) D. Bohm and E. P. Gross, Phys. Rev. 75, 1851, 1864 (1949).

7) A. Vlasov, J. Phys. U.S.S.R. 9, 25, 130 (1945).

8) I. Langmuir, Proc. Nat. Acad. Sci. U.S. 14, 627 (1928).

Processes involving characteristic lengths longer than the Debye length call into play the co-operative behavior of the particles of an ionized gas. Vlasov assumed that apart from close collisions, a particle moves in an average electric field arising from the other particles and proposed to describe the system by a one-particle distribution function $f(\vec{v}, \vec{x}, t)$. He wrote for the description of longitudinal disturbances,

$$\frac{\partial f}{\partial t} + (\vec{v} \cdot \vec{\nabla}) f - \frac{e\vec{E}}{m} \cdot \frac{\partial f}{\partial v} = \frac{\delta f}{\delta t}\bigg|_c \qquad , \qquad (1)$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi e \left\{ n_o - \int f \, d\vec{v} \right\} \qquad , \qquad (2)$$

where n_o is the density of the uniform continuous positive background. The force $e\vec{E}$ arising from the charge concentrations of the particles themselves has been inserted in the position in the kinetic equation generally reserved for body forces. Vlasov termed this the integral interaction term; it is designed to treat the forces on a given particle arising from other particles at distances greater than a Debye length. The term $\frac{\delta f}{\delta t}\big|_c$ represents the instantaneous change of the distribution function arising from closer collisions. The question then arises as to the explicit form to take for $\frac{\delta f}{\delta t}\big|_c$. The very close encounters at a distance less than the interparticle distance can be treated by the binary collision term of Boltzmann type. For the more distant collisions such an approach is less certain. It is important to note that in a low-pressure plasma the Debye length is, in general, many times the interparticle distance. Thus, in a collision between two particles there are, in general, many other particles between them. In spite of this, many authors [9] have treated various processes in ionized gases using the binary collision approach even for these distant

9) D. Gabor, Z. Physik 84, 474 (1933); L. Landau, Physik. Z. Sowjetunion 10, 154 (1936); D. Bohm and L. H. Aller, Astrophys. J. 105, 131 (1947); R. Cohen, L. Spitzer, Jr., and P. McR. Routly, Phys. Rev. 80, 230 (1950).

collisions. They treat the distant collisions as involving small momentum transfers and expand the distribution functions in the collision integral term in powers of the momentum transfers. Terms very similar to those given by the Fokker-Planck equation are obtained. This procedure is, of course, approximate since it attempts to replace many body collisions by a series of binary collisions. The equation neglects the "Onsager correction" and other correlation effects; these require a more refined description in terms of many particle distribution functions. However, in the present primitive state of the theory it is worthwhile exploring the consequences of this approach. In a later paper we shall discuss the application of this method to the plasma oscillation problem in some detail. The Fokker-Planck approach has already been used in the study of plasma oscillations by Logunov.[10] He obtains a damping and dispersion, but his method is incapable of treating high-pressure plasmas because of an inadequate representation of the collision terms.

3. KINETIC MODELS OF COLLISION PROCESSES

The equations described above for neutral and ionized gases are complicated to handle in practice because of the intractable nature of the Boltzmann binary collision term. We shall now discuss some simple kinetic models which permit of exact mathematical treatment including the solution of definite boundary-value problems. A more detailed discussion of the validity of our representation of collisions and its relation to the standard Boltzmann equation will be given in a subsequent paper. In this section of the present paper we introduce the simplest kinetic model; we give the underlying motivation for the form of this model from one point of view.

In many other kinetic problems, for example, the effect of electron collisions on the propagation of radio waves in the ionosphere, it is convenient to

10) A. A. Logunov, Zhur. Eksptl. Teoret. Fiz. 20, 458 (1950).

avoid the complexities of the Boltzmann equation by using a mean free-path treatment. One replaces the collision integral by a relaxation term of the form

$$\frac{\delta f}{\delta t}\bigg|_c = \frac{f_o - f(\vec{v}, \vec{x}, t)}{\tau(v)} \qquad , \qquad (3)$$

where $\tau(v)$ is a velocity dependent collision time. This expresses the fact that collisions tend to relax the distribution function to an equilibrium value f_o. We illustrate our discussion of collision models by referring to oscillatory problems where a characteristic time enters in a natural way. The relaxation model then describes the destruction of phase of an ordered motion on collision and leads to a damping frequency of order $1/\tau$ in the amplitude, where τ is some suitable average collision time. This type of model has the defect[11] that charge is not conserved instantaneously but only over a cycle. It is, however, easy to remedy this at least in the case of constant collision time by taking for the collision term

$$\frac{\delta f}{\delta t}\bigg|_c = - \frac{f}{\tau} + \frac{n(\vec{x}, t)}{\tau} f_o(\vec{v}) \qquad , \qquad (4)$$

where $n(\vec{x}, t) = \int f \, d\vec{v}$ is the fluctuating density. Thus particles in a range $d\vec{v}$ about velocity $\vec{v}$ are absorbed at a rate proportional to the number $f(\vec{v}, \vec{x}, t)$ at $(\vec{x}, t)$ and reemitted at a rate proportional to the density at $(\vec{x}, t)$ and with a Maxwellian distribution of velocities.[*] We may regard this as a model representing electron neutral collisions. (This representation will be studied in more detail in Paper 2, where we treat the coupling of charged and uncharged systems more adequately.) Upon integrating over velocities we find $\int \frac{\delta f}{\delta t}\bigg|_c d\vec{v} = 0$ so that the number of particles is instantaneously conserved in collisions in contrast to the earlier case where the conservation holds only when averaged over finite times. There is, however, no instantaneous conservation of momentum or

11) E. P. Gross, Phys. Rev. 82, 232 (1951).

* For a discussion of this model from a different point of view, Ref. 6, p. 1864.

energy. In fact, we have $\int m \vec{v} \frac{\delta f}{\delta t}\bigg|_c d\vec{v} = -\frac{m}{\tau} \int \vec{v} f d\vec{v}$, $\int \frac{mv^2}{2} \frac{\delta f}{\delta t}\bigg|_c d\vec{v} = \frac{m}{2\tau} \left\{ \frac{3kT_o}{m} - \int v^2 f d\vec{v} \right\}$. This represents a gain or loss of total energy and momentum during collision, depending on the phase of f.

Thus far we have discussed a model which can represent only electron-neutral or ion-neutral collisions in which the collision term is linear in the electronic distribution function. The linear feature would also be present in a more exact study of the collisions. However, in considering electron-electron or ion-ion collisions which are important in the high-pressure ionized gas, we will expect to find collision terms quadratic in the electron and the ion distribution functions. (In the study of small amplitude disturbances such terms may be linearized, but we now seek a general formulation.) We shall construct a simple model, quadratic in f which reproduces essential features of the collisions without introducing the troublesome Boltzmann collision integral. Let us write

$$\frac{\partial f}{\partial t} + (\vec{v} \cdot \vec{\nabla}) f - \frac{e\vec{E}}{m} \cdot \frac{\partial f}{\partial \vec{v}} = -\frac{n(\vec{x}, t)}{\sigma} f(\vec{v}, \vec{x}, t) + \frac{n^2(\vec{x}, t)}{\sigma} \bar{F} \quad (5)$$

$$\bar{F} = \left(\frac{m}{2\pi kT_o} \right)^{3/2} e^{-\frac{mv^2}{2kT_o}} . \quad (6)$$

Thus $\int \bar{F} d\vec{v} = 1$. We have taken the rate at which particles leave a particular velocity range at $(\vec{x}, t)$ as proportional to $f(\vec{v}, \vec{x}, t)$ and to the density $n(\vec{x}, t)$. The number of particles emitted at $(\vec{x}, t)$ equals the number absorbed at $(\vec{x}, t)$; this number is proportional to the square of the density and is emitted with a Maxwellian velocity distribution. $\lambda = n_o/\sigma$ is a collision frequency, here assumed independent of velocity. It is easily verified that the total number of particles is instantaneously conserved by the collision processes, while momentum and energy are not.

It is now worthwhile to examine the dispersion relation for small-amplitude oscillatory processes governed by Eq. (5). For brevity, we adopt the

approach of looking for plane-wave solutions of the type exp $(i (px - \omega t))$. We shall show later that this procedure is valid for the treatment of certain types of initial value problems for finite collision time and for waves of large wavelength. We write

$$f = f_o (1 + \phi) \qquad f_o = n_o \bar{F} \qquad , \qquad (7)$$

$$n (\vec{x}, t) = n_o (1 + \nu) \qquad , \qquad (8)$$

where ϕ and ν are dimensionless quantities small compared to unity. We assume ϕ, ν, and E oscillate as exp $(i (px - \omega t))$ and neglect products of these variables. Then Eq. (5) becomes

$$i (pu - \omega) \phi + \frac{eE}{kT_o} u = \lambda (\nu - \phi) \qquad . \qquad (9)$$

We also have

$$\nu = \int \bar{F} \phi \, d\vec{v} \qquad , \qquad (10)$$

$$E = - \frac{4\pi e n_o}{ip} \nu \qquad . \qquad (11)$$

Eliminating E we find the distribution function

$$\phi = \text{Re} \frac{\nu}{\lambda + i (pu - \omega)} \left\{ \lambda + \frac{4\pi n_o e^2}{kT_o} \frac{u}{ip} \right\} \qquad , \qquad (12)$$

and the dispersion relation

$$1 = \int \frac{\bar{F} \, d\vec{v}}{\lambda + i (pu - \omega)} \left\{ \lambda + \frac{4\pi n_o e^2}{kT_o} \frac{u}{ip} \right\} \qquad . \qquad (13)$$

If one expands in powers of $ipu/(\lambda - i\omega)$ and restricts oneself to the study of long wavelength one finds

$$\omega^2 = \omega_p^2 - i\omega\lambda - \frac{kT_o}{m} p^2 \left\{ \frac{3\omega_p^2}{(\lambda - i\omega)^2} - \frac{\lambda}{\lambda - i\omega} \right\} \qquad . \qquad (14)$$

We conclude from this relation that the oscillations are strongly damped at high pressures. For the case of a charged gas the complex frequency is approximately $\omega \simeq - \frac{i\lambda}{2} \pm \left(- \frac{\lambda^2}{4} + \omega_p^2 \right)^{1/2}$; this is of the order of the collision frequency for λ small and always corresponds to damping. For a neutral gas, $\omega_p = (\frac{4\pi n_o e^2}{m})^{1/2} = 0$, and so $\omega^2 \simeq - i\omega\lambda + \frac{kT_o}{m} p^2 \frac{\lambda}{\lambda - i\omega}$; thus in the limit of very high collision frequency one obtains highly damped oscillations instead of undamped sound waves.

We thus see that this model is incapable of yielding correct results at the high-pressure limit? What is the origin of this failure. As discussed earlier and in Ref. 6, the processes of organization are very different for a plasma and for a sound wave. In particular, one must recognize the velocity organization characteristic of a sound wave. Thus a particle with excess velocity at a point makes collisions which slow it down to the flow velocity at the point in question. Similarly particles which are moving too slowly tend to have their speed increased to that of the flow velocity. The collision models studied thus far fail to incorporate this feature so that it is not surprising that they do not yield sound-type oscillations in the high-density limit.

We now propose a simple model which succeeds in giving an adequate description of both the high- and low-density plasma. For a one-component system, one writes

$$\frac{\partial f}{\partial t} + (\vec{v} \cdot \vec{\nabla}) f - \frac{e\vec{E}}{m} \cdot \frac{\partial f}{\partial \vec{v}} = - \frac{n(\vec{x}, t)}{\sigma} f(\vec{v}, \vec{x}, t) + \frac{n^2}{\sigma} \Phi \qquad , \qquad (15)$$

$$\Phi = (\frac{m}{2\pi k T(\vec{x}, t)})^{3/2} \exp \left[- \frac{m}{2kT(\vec{x}, t)} (\vec{v} - \vec{q}(\vec{x}, t)^2 \right] \qquad . \qquad (16)$$

$$\int \Phi \, d\vec{v} = 1$$

$$n(\vec{x}, t) = \int f(\vec{v}, \vec{x}, t) \, d\vec{v} \qquad (17)$$

$$q(\vec{x}, t) = \frac{1}{n(\vec{x}, t)} \int \vec{v} \, f(\vec{v}, \vec{x}, t) \, d\vec{v} \qquad (18)$$

$$\frac{3kT(\vec{x}, t)}{m} = \frac{1}{n} \int (\vec{v} - \vec{q})^2 \, f \, d\vec{v} \qquad (19)$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi e \left\{ n_o - \int f \, d\vec{v} \right\} \qquad . \qquad (20)$$

$\vec{q}(\vec{x}, t)$ and $T(\vec{x}, t)$ define the flow velocity and temperature at $\vec{x}$ and t. We have assumed the reemitted particles at $(\vec{x}, t)$ to emerge with a Maxwellian distribution centered about the flow velocity $\vec{q}$ and for a temperature $T(\vec{x}, t)$. It is easily verified that, with the definitions of $\vec{q}$ and T given, the collision terms

instantaneously conserve particle number, momentum, and energy. The kinetic equation (15) is a nonlinear integro-partial differential equation. However, it is considerably simpler than the standard Boltzmann equation since the distribution function enters into the collision terms of Eq. (15) in a simple way; as f, $\int f\, d\vec{v}$, $\int \vec{v}\, f\, d\vec{v}$ and $\int v^2\, f\, d\vec{v}$. Thus one finds a solution of Eq. (15) in terms of the undetermined functions $\vec{q}(\vec{x}, t)$, $n(\vec{x}, t)$, $T(x, t)$, $E(x, t)$, and then inserts this solution into Eqs. (17) to (20). This establishes the conditions of compatibility which serve to determine the solution completely. For the particular case of small amplitude oscillations the procedure yields a dispersion relation specifying the connection between frequency and wavelength in its dependence on collision frequency, plasma frequency, temperature, etc.

The use of conditions of compatibility bears a certain resemblance to the first step in the Enskog-Chapman method of solving the Boltzmann equation. However, the methods are quite different. In the E-C technique one assumes a locally Maxwellian form for the distribution function in first approximation and then determines the density, velocity and temperature entering in this choice by conditions of compatibility; here we change the collision term itself and solve for the distribution function rigorously. The solution will, in general, not have the form assumed in the Enskog-Chapman method. Our approach has the virtue that one can solve definite initial and boundary-value problems wholly within the framework of a microscopic formalism. The model we have described involves the assumption of a collision time independent of velocity. In Paper 4 we shall show that this is not an essential restriction and shall discuss the generalizations to take into account velocity-dependent collision times as well as others; in the present paper we shall follow the consequences of Eqs. (15) to (20) inasmuch as it appears to be the simplest model capable of yielding physically satisfactory results throughout the entire range of pressures.

There exists a number of studies of the oscillations of ionized gases using transport equations. For example, the Thomsons[12] and subsequently Bailey[13] base their discussions on Maxwell's equations of transfer. As applied to a one-component system of charged particles these equations are those of continuity, of momentum transfer and Poisson's equation. When no static magnetic field is present one obtains the dispersion relation $\omega^2 = \omega_p^2 + (\frac{kT_o}{m})p^2$. It is not clear from these transport treatments whether the results are meant to apply at low pressures or high pressures; in fact, they could be valid only at high pressures. Even there, however, they are inexact, as they correspond to an isothermal treatment. Thus, in terms of the model discussed above, the temperature T is treated as a constant and energy balance is not maintained instantaneously by collisions; we shall obtain Thomson's results from our model at high pressures in the isothermal approximation. The high-pressure nature of the Thomson approach for a one-component system is brought out in an alternative treatment by Linder[14] who adds a pressure term to the equation of motion of a particle and with an isothermal assumption obtains the same result as Thomson. At low pressures the correct dispersion relation as found by an exact treatment of the kinetic equation is $\omega^2 = \omega_p^2 + \frac{3kT_o}{m}p^2$, differing from that at high pressures by the factor 3. We expect then that, in an isothermal treatment, there will be a change in the frequency corresponding to a given wavelength as one goes from low to high pressures. This is indeed the case as will be shown with our collision model; in addition, we find an accompanying absorption which reaches a maximum value when the collision frequency is of the order of magnitude of the plasma frequency.

12) J. J. and G. P. Thomson, "Conduction of Electricity through Gases," Cambridge University Press, London, 1933, Vol. 2, p. 353.

13) V. A. Bailey, Phys. Rev. 78, 428 (1950).

14) E. G. Linder, Phys. Rev. 49, 753 (1936).

We have thus far confined discussion of the transport method to iso-
thermal models. It is possible to generalize the transport procedure to in-
clude temperature variations by applying a moment method to the Boltzmann
equation. [11] One finds a dispersion relation $\omega^2 = \omega_p^2 + \frac{5}{3} p^2 kT_0/m$. For the
uncharged case this gives the correct "adiabatic" law of sound-wave propa-
gation $\omega = (5/3)^{1/2} (kT_0/m)^{1/2} p$. We shall see that the result is also correct
for an ionized gas; the transport approach owes its validity at high pressures
to the fact that momentum and energy are collisional invariants. Existing
transport treatments are certainly incorrect quantitatively in the low-pressure
region and even fail to give a qualitatively adequate description when magnetic
fields are present. [11] For the treatment of the effects of heat conduction and
viscosity which depend on the mean free path a kinetic approach of the type
here presented is needed.

4. ISOTHERMAL APPROXIMATION

The simplest collision model yielding both low-pressure plasma waves
and also sound waves at high pressures may be obtained by taking T constant in
Eqs. (15) to (20). This is not entirely realistic since it yields the Newtonian
isothermal sound speed at high pressures. But the fact that the number of vari-
ables is one less, simplifies the calculations considerably. For the sake of
ease of exposition we shall discuss this case in detail and only sketch the treat-
ment of the more general case including temperature variations. The isothermal
case is also of some interest in itself since the work of Thomas and Bailey is
based on the isothermal approximation. The results of main physical interest
are presented in Figs. 1 to 5. Figures 1 and 2 show the dispersion and absorption
of a one-component charged system as a function of collision time. Figure 5 shows
the absorption and the translational dispersion effect in a neutral monatomic gas.

The equations describing the system of particles in the approximation of

this section are Eqs. (15) to (18), (20). We do not use Eq. (19) but set the temperature equal to a constant T_o, the equilibrium temperature of the gas. As we are concerned with small amplitude oscillations we shall neglect quantities quadratic in the oscillating variables and arrive at a set of linear equations. The consistency of the linearization procedure can be checked by comparing the terms neglected (as estimated with the linear approximation) with the terms retained. For the case of low-pressure plasma oscillations a breakdown of the linear approximation is closely connected with the presence of a large number of trapped particles moving with a velocity close to the wave velocity. These particles play a role quite different from other particles and make possible types of solutions outside the scope of a linear approximation.

Put

$$f(\vec{v}, \vec{x}, t) = n_o \bar{F} \left\{ 1 + \phi \right\} \tag{21}$$

$$n(\vec{x}, t) = n_o \left\{ 1 + \nu \right\} \ . \tag{22}$$

Then ϕ and ν are dimensionless quantities. Equations (15) to (20) then become

$$\nu(\vec{x}, t) = \int \phi \, \bar{F} \, d\vec{v} \tag{23}$$

$$\vec{q}(\vec{x}, t) = \int \vec{v} \, \phi \, \bar{F} \, d\vec{v} \tag{24}$$

$$\vec{\nabla} \cdot \vec{E} = -4\pi e \, n_o \nu \ . \tag{25}$$

In the linearized theory, ϕ, ν, $\vec{q}$, $\vec{E}$ are quantities of first order of smallness. We consistently neglect all products and higher powers of these quantities. On expanding the exponential in Eq. (16) (neglecting q^2 and higher powers), we find

$$\Phi = \bar{F} \left\{ 1 + \frac{m}{kT_o} \vec{v} \cdot \vec{q} \right\} \ . \tag{26}$$

The linearized kinetic equation then becomes

$$\frac{\partial \phi}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \phi + \frac{e\vec{E} \cdot \vec{v}}{kT_o} = \lambda \left(\nu - \phi + \frac{m}{kT_o} \vec{v} \cdot \vec{q} \right) \ . \tag{27}$$

Equations (21) to (27) form the basis for the further analysis of this section.

Landau[15] has emphasized the lack of rigor in the procedure of looking for

15) L. D. Landau, J. Phys. U.S.S.R. 10, 25 (1946).

solutions of Eq. (27) of the type where all quantities have a time and space variation $e^{i(px - \omega t)}$. One must consider also the solutions of the homogeneous parts of Eqs. (20) and (27) and take the correct linear combination in order to solve a definite initial or boundary-value problem. In this paper we examine a one-dimensional initial value problem, the propagation of a pulse. More general boundary-value problems will be treated in subsequent papers. Suppose that at times $t < 0$ the particle density is uniformly n_o throughout unbounded space except for a small region between $-x_o$ and x_o confined between two walls, where the density is $n_o + \nu_o$. At time $t = 0$ the walls are removed and we ask about the subsequent behavior of the system. To solve the problem the distribution function at $t = 0$ must be specified. This initial distribution depends on the particular (macroscopic) mode of preparation of the system. If, for example, the present system has been prepared by compressing the gas in $(-x_o, +x_o)$ slowly over a time very long compared to the mean time between collisions, the initial distribution function will be very nearly Maxwellian. For the case of sound waves it is clear what will happen. Two pulses are propagated with sound speed in opposite directions parallel to the x axis; the shape of each pulse is maintained except for the small absorption and dispersion effects. In a low-pressure, ionized gas the pulse will move slowly but will alter its shape appreciably because of the high dispersion.

We take as initial conditions at $t = 0$

$$
\left.
\begin{array}{l}
\left.
\begin{array}{ll}
\phi(\vec{v}, \vec{x}, 0) = \nu_o \, \overline{F}(\vec{v}) & \text{for } -x_o < x < x_o \\
\qquad\qquad\quad = 0 & \text{otherwise}
\end{array}
\right\} \\[2ex]
\left.
\begin{array}{ll}
\nu = \nu_o & \text{for } -x_o < x < x_o \\
\quad = 0 & \text{otherwise}
\end{array}
\right\}
\end{array}
\right\}
\tag{28}
$$

The mass velocity $\vec{q}$ is zero throughout all space at $t = 0$.

The special problem is typical of a large class of problems which may be

solved by first performing a Fourier expansion in unbounded x space. We write

$$\phi(\vec{v}, \vec{x}, t) = \int \phi_{\vec{p}}(\vec{v}, t) e^{i\vec{p} \cdot \vec{x}} d\vec{p} \qquad \text{, etc.} \qquad (29)$$

From Eqs. (23) to (27) we find the following equations for the Fourier transforms,

$$\frac{\partial \phi_{\vec{p}}}{\partial t} + i\vec{p} \cdot \vec{v} \phi_{\vec{p}} + \frac{e\vec{E}_{\vec{p}} \cdot \vec{v}}{kT_o} = \lambda(\nu_{\vec{p}} - \phi_{\vec{p}} + \frac{m}{kT_o} \vec{v} \cdot \vec{q}_{\vec{p}}) \qquad (30)$$

$$\nu_{\vec{p}}(t) = \int \phi_{\vec{p}}(\vec{v}, t) \bar{F} d\vec{v} \qquad (31)$$

$$\vec{q}_{\vec{p}}(t) = \int \vec{v} \phi_{\vec{p}} \bar{F} d\vec{v} \qquad (32)$$

$$i\vec{p} \cdot \vec{E}_{\vec{p}} = -4\pi e n_o \nu_{\vec{p}}(t) \qquad . \qquad (33)$$

These equations are a determined set if one specifies $\phi_{\vec{p}}(\vec{v}, t)$ at time $t = 0$.

Thus we must find the solutions of Eqs. (30) to (33) for which each Fourier component satisfies the appropriate initial condition. One then superposes the $\phi_{\vec{p}}$ to find $\phi(\vec{v}, \vec{x}, t)$.

For example, in the problem discussed above

$$\left.\begin{array}{l} \phi_{\vec{p}}(\vec{v}, 0) = \int_{-x_o}^{+x_o} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \nu_o \bar{F} e^{-i\vec{p} \cdot \vec{x}} d\vec{x} \\[12pt] = 2 \nu_o \dfrac{\sin p_x x_o}{p_x} \delta(p_y) \delta(p_z) \bar{F} \\[12pt] \nu_{\vec{p}}(0) = 2 \nu_o \dfrac{\sin p_x x_o}{p_x} \delta(p_y) \delta(p_z) \end{array}\right\} \qquad (34)$$

where $\delta(x)$ is the Dirac delta function. Let us suppose that $\nu_{\vec{p}}(t)$ has the value $\nu_{\vec{p}}(t) = \dfrac{\nu_{\vec{p}}(0)}{2}(e^{+i\omega t} + e^{-i\omega t})$. For oscillations of a one-component plasma without thermal motions, $\omega = \omega_p$ is independent of p and the disturbance remains localized. For isothermal sound waves, $\omega = p\left(\dfrac{kT_o}{m}\right)^{1/2}$ and we obtain two oppositely directed pulses. For long wavelengths and low pressures, the plasma, including thermal effects, obeys the dispersion relation $\omega \simeq \omega_p + \dfrac{3}{2} \dfrac{kT_o}{m} \dfrac{p^2}{\omega_p}$.

To investigate the properties of a density pulse we form

$$\nu(\vec{x}, t) = \nu_o \int e^{i\vec{p}\cdot x} \frac{\sin p_x x_o}{p_x} \delta(p_y) \delta(p_z) \left[e^{i\omega_p t} e^{-i\frac{3}{2}\frac{kT_o}{m}p^2 t} + \text{conj. comp.} \right].$$

The detailed discussion of this integral is complicated and will not be given here. In the limit of long times, $\nu(\vec{x}, t)$ tends to zero throughout all space indicating the spreading of the initially sharp pulse due to dispersion. As we shall see later, in reality the dependence of $\nu_p(t)$ is more complicated than that assumed here; for pulses with widths large compared to the Debye length the general features are unaltered.

We solve Eqs. (30) to (33), following Landau's procedure, by performing a Laplace transformation with respect to the time variable.
We write

$$\phi_{p,\sigma} = \int_o^\infty \phi_p(\vec{v}, t) e^{-\sigma t} dt \tag{35}$$

for the image of $\phi_p(\vec{v}, t)$. For the inverse transformation we have

$$\phi_p(\vec{v}, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi_{p,\sigma} e^{\sigma t} d\sigma , \tag{36}$$

where the integration is carried out along a line in the right half plane of the complex variable σ. For the case of waves propagated along the x axis Eqs. (30) to (33) become

$$(\lambda + \sigma + ipu) \phi_{p,\sigma} = -\frac{eu}{kT_o}(E_{p,\sigma})_x + \lambda \nu_{p,\sigma} + \lambda \frac{m}{kT_o} u(q_{p,\sigma})_x + \phi_p(\vec{v}, 0) \tag{37}$$

$$\nu_{p,\sigma} = \int \phi_{p,\sigma} \bar{F} d\vec{v} \tag{38}$$

$$(q_{p,\sigma})_x = \int u \phi_{p,\sigma} \bar{F} d\vec{v} \tag{39}$$

$$ip(E_{p,\sigma})_x = -4\pi e n_o \nu_{p,\sigma} . \tag{40}$$

Here $\phi_p(\vec{v}, 0)$ is the Fourier transform of the initial velocity distribution. By multiplying Eq. (37) by $\bar{F}$ and integrating over all velocities one finds

$$\sigma \nu_{p,\sigma} + ip(q_{p,\sigma})_x = \nu_p(0) , \tag{41}$$

where $\nu_{\vec{p}}(0) = \int \phi_{\vec{p}}(\vec{v}, 0) \, \bar{F} \, d\vec{v}$,

Using this equation together with Eq. (40), we eliminate $(q_{\vec{p}, \sigma})_x$ and $(E_{\vec{p}, \sigma})_x$

from Eq. (37) and are left with the set of equations

$$(\lambda + \sigma + ipu) \, \phi_{\vec{p}, \sigma} = (\lambda - \frac{\lambda mu}{kT_o} \frac{\sigma}{ip} + \frac{k_o^2}{ip} u) \, \nu_{\vec{p}, \sigma} + \phi_{\vec{p}}(0) + \frac{\lambda mu}{kT_o} \frac{\nu_{\vec{p}}(0)}{ip} \quad (42)$$

$$\nu_{\vec{p}, \sigma} = \int \phi_{\vec{p}, \sigma} \, \bar{F} \, d\vec{v} \quad . \quad (43)$$

Solving for $\nu_{\vec{p}, \sigma}$ we find

$$\nu_{\vec{p}, \sigma} = \frac{\int \bar{F} \, d\vec{v} \, (\frac{\lambda m}{kT_o} \frac{u \, \nu_{\vec{p}}(0)}{ip} + \phi_{\vec{p}}(0))}{1 - \int \frac{\bar{F} \, d\vec{v}}{\lambda + \sigma + ipu} \left\{ \lambda - \frac{\lambda m}{kT_o} \frac{u\sigma}{ip} + \frac{k_o^2}{ip} u \right\}} \quad . \quad (44)$$

Here $k_o^2 = \omega_p^2/a^2$. The expression for $\phi_{\vec{p}, \sigma}$ is obtained by inserting this value of

$\nu_{\vec{p}, \sigma}$ into Eq. (42). Equations (42) and (44) thus provide a complete solution to

the initial-value problem since we have merely to invert the Laplace transform

to find the dependence of the density and velocity distribution on time. However,

the integrations are along paths in the right half plane of the complex variable σ

and, aside from being difficult to evaluate, yield little physical insight into the

behavior of the solution. We therefore note that Eqs. (42) and (44) are defined

only for σ in the right half complex plane but that it is possible to effect an

analytic continuation of the functions into the left half plane of σ by modifying the

path of integration in the integrals with respect to velocity. We integrate in the

complex u plane along a path C which passes below the singularity, $\lambda + \sigma + ipu = 0$

or $u = - \frac{(\lambda + \sigma)}{ip}$. The two integrals entering in the numerator and denominator

of Eqs. (44) are then entire functions in the complex σ plane.[15]

With the path of integration C, the Laplace transform of the density $\nu_{\vec{p}, \sigma}$

will have poles at values of σ given implicitly by

$$1 - \int_C \frac{\bar{F} \, d\vec{v}}{\lambda + \sigma + ipu} \left\{ \lambda - \frac{\lambda m}{kT_o} \frac{u\sigma}{ip} + \frac{k_o^2}{ip} u \right\} = 0 \quad (45)$$

There will be an infinite number of such poles, all lying in the left half plane of σ. If one deforms the contour into the left half plane in Eq. (45), one finds that the density after a long time t is given very nearly in terms of the poles at the residues of $\nu_{\vec{p}}, \sigma$. For shorter times the remaining parts of the contour give a significant contribution and there is in fact no advantage in deforming the contour in the above way. For t large then

$$\nu(t) = \int \bar{F} \, d\vec{v} \left(\frac{\lambda m}{kT_o} \frac{u \, \nu_{\vec{p}}(0)}{ip} + \phi_{\vec{p}}(0) \right) \sum_j e^{\sigma_j t} \quad ,$$

since the numerator in Eq. (44) is independent of σ. $\nu(t)$ may thus be represented approximately as a sum of damped oscillating terms. The asymptotic behavior for t large is determined by that root σ_1 of the dispersion relation Eq. (45) which has the largest real part.

The dispersion relation Eq. (45) differs from that obtained by the $e^{i(px-\omega t)}$ substitution in that the path of integration is not necessarily along the real axis. We shall see shortly, however, that the substitution analysis gives a correct account of the behavior at long times when the damping is not very great.

The behavior of the velocity distribution $\phi_{\vec{p}}(\vec{v}, t)$ is different from that of the density $\nu_{\vec{p}}(t)$ since the Laplace transform of $\phi_{\vec{p}, \sigma}$ has an additional pole at a value of σ given by

$$\lambda + \sigma_o + ipu = 0 . \tag{46}$$

In this expression the velocity u is, of course, real since we are investigating the behavior of the distribution function for particles of a given real velocity u. The asymptotic behavior of $\phi(\vec{v}, t)$ depends essentially on the relative values of the real parts of the pole given by $\sigma_o = -ipu - \lambda$ and of the pole σ_1 with largest real part satisfying the dispersion relation Eq. (45). It is easy to see that the dispersion relation will be the determining factor when the damping frequency is smaller than the collision frequency λ. For this case both the density and the distribution function (for any value of the velocity) have the same time dependence

$e^{\sigma_1 t}$. We shall call this the "collision-damping" case. From our later expression Eq. (68) for the frequency as a function of wavelength and collision time, we see that $\text{Re } \sigma_1 = \omega_i = -\left[\lambda/(\lambda^2 + 1)\right]\left[p^2 a^2/\omega_p^2\right]$ for waves larger than either the Debye length or the mean free path. We see that the damping frequency is indeed smaller than the collision frequency for these waves. This condition has the further consequence that the contour C may be taken along the real axis so that the $e^{i(px - \omega t)}$ substitution analysis will lead to correct results for the behavior at large values of time.

If, however, one is dealing with processes in which wavelengths less than both the Debye length and the mean free path are important, the path C cannot be deformed into the path along the real axis; this is due to the fact that the damping frequency is higher than the collision frequency. The distribution function at long times will vary as $e^{-\lambda t} e^{-iput}$ while the density will have the more rapid decay given by the root σ_1 of the Eq. (45). Thus the distribution function for long times contains, in addition to the organized component of velocity, "free-particle" components which give rise to negligible macroscopic density fluctuations.

For the "collision-damping" case one can easily write explicit expressions for the asymptotic behavior of the oscillating quantities. We have

$$\nu = \nu_0 \text{ Re } e^{i(px - \omega t)} = \nu_0 e^{\omega_i t} \cos(px - \omega_r t) \tag{47}$$

$$E_x = -\frac{4\pi e n_0}{p} e^{\omega_i t} \sin(px - \omega_r t) \tag{48}$$

$$\phi = \frac{\nu_0 e^{\omega_i t}}{(\lambda+\omega_i)^2+(pu-\omega_r)^2}\left[\cos(px-\omega_r t)\left\{(\lambda+\omega_i)(1+\frac{u\omega_r}{pa^2})\lambda - (pu-\omega_r)(1-\frac{\lambda\omega_i}{\omega_p^2})\frac{u\omega_p^2}{pa^2}\right\}\right.$$
$$\left. + \sin(px-\omega_r t)\left\{\lambda(pu-\omega_r)\left(1+\frac{u\omega_r}{pa^2}\right) - (\lambda+\omega_i)\left(1-\frac{\lambda\omega_i}{\omega_p^2}\right)\frac{u\omega_p^2}{pa^2}\right\}\right] \tag{49}$$

The real and imaginary parts of ω are given by the equations obtained in our study of the dispersion relation in Secs. 5 to 7.

Using the above expressions it is now easy to verify that the approximations made in the linearization procedure are valid for all velocities in this long wavelength case. The amplitude ν_0 can be chosen arbitrarily small and, provided the collision frequency is not zero, the linearization leads to consistent results.

For the high-pressure limit we have from Eq. (68) $\omega_r \rightarrow \omega_p + \frac{1}{2} \frac{p^2 a^2}{\omega_p^2}$; $\omega_i \rightarrow 0$ as $\lambda \rightarrow \infty$. The distribution function is

$$\phi \rightarrow \nu_0 \cos (px - \omega_r t) \left(1 + \frac{u}{a^2} \frac{\omega_r}{p}\right) \qquad . \qquad (50)$$

For the low-pressure plasma $\omega_r \rightarrow \omega_p + \frac{3}{2} \frac{p^2 a^2}{\omega_p^2}$; $\omega_i \rightarrow -\frac{\lambda}{\omega_p} \frac{p^2 a^2}{\omega_p^2}$ as $\lambda \rightarrow 0$. Then

$$\phi \rightarrow \frac{\nu_0 e^{\omega_i t}}{\lambda^2 + (pu - \omega_r)^2} \left\{ -\cos(px - \omega_r t) \frac{u\omega_p^2}{pa^2} (pu - \omega_r) - \sin(px - \omega_r t) \, 2 \, \frac{\lambda u \omega_p^2}{pa^2} \right\} \quad . \qquad (51)$$

For sound waves in an uncharged gas we again have $\phi \rightarrow \nu_0 \cos (px - \omega_r t)\left(1 + \frac{u}{a^2} \frac{\omega_r}{p}\right)$; here however, one must take $\omega_r = pa$. Thus

$$\phi \rightarrow \nu_0 \cos p (x - at) (1 + \frac{u}{a}) \qquad . \qquad (52)$$

All the above distribution functions exhibit the same general feature that the magnitude is greater for those particles moving in the direction of the wave than it is for those moving in the opposite direction. This is in fact necessary for maintaining a wave. For the low-pressure plasma case we note also the opposite contributions from particles moving faster and slower than the wave.

5. DISPERSION RELATION FOR INITIAL VALUE PROBLEM

The present section is devoted to a discussion of the techniques needed for the study of the dispersion relation Eq. (45). The results for important special cases are given in the following two sections. At this stage, it is con-

venient to introduce dimensionless variables. We measure frequencies in units of the plasma frequency ω_p, velocities in units of the thermal velocity $a = (kT_0/m)^{1/2}$, and lengths in units of the Debye length $\ell_D = (kT/4\pi n_0 e^2)^{1/2}$. In these units the dispersion relation becomes

$$ip = \frac{1}{(2\pi)^{1/2}} \int_c \frac{d\zeta \, e^{-\zeta^2/2}}{\lambda + \sigma + ip\zeta} \left\{ \zeta + ip\lambda - \sigma\lambda\zeta \right\} \, d\zeta \quad . \tag{53}$$

We shall at times deal with ω rather than σ; the two quantities are connected by $\sigma = -i\omega$.

Let us first study the position of the singularity of the integrand in the ζ plane. Since the oscillations are damped, σ has a negative real part and a negative imaginary part. Thus $\lambda + \sigma$ can be either positive or negative according as the damping frequency is less than or greater than the collision frequency. Since p is real this means that the singularity $\zeta_0 = -(\lambda + \sigma)/ip$ is in either the first or the fourth quadrant in the ζ plane. For ζ_0 in the first quadrant, the contour appropriate to the $e^{i(px - \omega t)}$ substitution, i.e., along the real axis, gives correct results. This represents the situation where the main damping arises from collisions. When ζ_0 is in the fourth quandrant, the damping frequency is larger than the collision frequency; the additional damping is of the Landau type and results from the destruction of phase by drift motion. The representation of this latter type of damping requires the use of the contour C. The damping frequency is small for long wavelengths but increases rapidly as the wavelength approaches the Debye length.

The most direct approach to the evaluation of Eq. (53) would appear to be an expansion of the term $1/(\lambda + \sigma + ip\zeta)$ in powers of $ip\zeta/(\lambda + \sigma)$ and an integration term by term. This is a development in powers of $p/(\lambda + \sigma)$ or, reverting to usual units, in powers of $pa/(\lambda - i\omega)$. Thus one requires pa to be small compared to either the collision frequency or the frequency of the waves. Alternatively, the wavelength is long compared to the Debye length or the mean free

path. This is actually the case in two important limits. The case of sound oscillations in neutral gases is one where the collision frequency is large compared to pa for all but very short waves and very low pressures. A second situation occurs with electron plasma oscillations of wavelength long compared to the Debye length ($\frac{pa}{\omega} \simeq \frac{pa}{\omega_p} = \frac{\text{Debye length}}{\text{wavelength}} \ll 1$) irrespective of the collision time, so that one can study the entire range of pressures. In fact, both of these cases can be treated by the same expansion. The expansion breaks down for heavily damped waves in both the neutral and ionized gas cases. An objection to the above procedure is that the series does not converge; it is, in fact, asymptotic so that it is useful where the first few terms are all important. In order to establish this and to give alternative developments, we express the dispersion relation in terms of the error function of complex argument.

Let us now displace the contour parallel to the real axis so that it has an imaginary co-ordinate equal to that of the singularity. If one breaks up the integral into the contribution of the small semicircle about the singularity and that of the principal value of the integral one finds

$$\frac{1}{(2\pi)^{1/2}} \int_c \frac{e^{-\zeta^2/2}}{\zeta - \theta} d\zeta = \frac{1}{(2\pi)^{1/2}} e^{-\theta^2/2} \left\{ i\pi - 2 \int_o^\infty e^{-\zeta^2/2} \frac{\sinh\theta}{\zeta} d\zeta \right\}, \quad (54)$$

where

$$\theta = \frac{\omega + i\lambda}{p} \quad . \quad (55)$$

Making use of the identity

$$\int_o^\infty e^{-\zeta^2/2} \frac{\sinh\theta}{\zeta} d\zeta = \left(\frac{\pi}{2}\right)^{1/2} \int_o^\theta d\theta' e^{\theta'^2/2} \quad , \quad (56)$$

and introducing

$$\Omega = -\frac{i\theta}{2} = \frac{\lambda - i\omega}{p\sqrt{2}} \quad , \quad (57)$$

we find the dispersion relation

$$-\frac{p^2}{1 + i\omega\lambda} = 1 + \left[\frac{ip\lambda}{1 + i\omega\lambda} + \frac{\omega + i\lambda}{p} \right] i\sqrt{2} \ F(\Omega) \quad . \quad (58)$$

Here

$$F(\Omega) = \left\{ \left(\frac{\pi}{2}\right)^{1/2} - \int_o^\Omega e^{-\Omega'^2} d\Omega' \right\} e^{\Omega^2} \quad (59)$$

is the error function of complex argument.

Examination of Ω shows that, for waves with positive real frequency, it lies either in the fourth or third quadrant. The function $F(\Omega)$ is treated in detail by Rosser.[16] It has the expansion

$$F(\Omega) = e^{\Omega^2} \left\{ \frac{\sqrt{\pi}}{2} - \sum_{n=0}^{\infty} (-1)^n \frac{\Omega^{2n+1}}{(2n+1)n!} \right\} \tag{60}$$

convergent for all values of Ω, but useful only for values of Ω with modulus less than unity. $F(\Omega)$ can also be expanded in an asymptotic series of the form

$$F(\Omega) = \frac{1}{2\Omega} - \frac{1}{4\Omega^3} + \frac{1 \cdot 3}{8\Omega^5} - \frac{1 \cdot 3 \cdot 5}{16\Omega^7} + \dots \tag{61}$$

This asymptotic series is in powers of $\frac{pa\sqrt{2}}{\lambda - i\omega}$ and yields the same result as the procedure outlined at the beginning of this section.

Finally, we note that $F(\Omega)$ can also be expanded as a continued fraction

$$F(\Omega) = \frac{1}{2\Omega} + \frac{2}{2\Omega} + \frac{4}{2\Omega} + \frac{6}{2\Omega} \dots \tag{62}$$

for Ω in the first and fourth quadrants. This expansion does not correspond to a simple development in powers of the Debye length or mean free path divided by the wavelength; however, it converges rapidly over a large portion of the complex Ω plane and is more generally useful than the asymptotic development. To obtain information in the regions of moderate and strong absorption for both neutral and ionized gases, we require a knowledge of $F(\Omega)$ over portions of the complex Ω plane not covered by any of the developments given; this knowledge has been obtained by numerical computations.

We also require a knowledge of $F(\Omega)$ for Ω in the second and third quadrants. This is found by making use of the identity

$$F(-\Omega) = \pi^{1/2} e^{\Omega^2} - F(\Omega) . \tag{63}$$

16) J. B. Rosser, "Theory and Application of $\int_0^z e^{-x^2} dx$," Mapleton House, Brooklyn, N.Y., 1948.

We see that the damping frequency is proportional to p^2, in contrast to the damping arising from the electron-neutral gas collisions, which contains a leading term independent of wavelength. The coefficient of $p^2 a^2 / \omega_p^2$ is plotted in Figs. 1 and 2. The absorption reaches a maximum when $\lambda = \omega_p$, i.e., when the collision frequency equals the plasma frequency. At this point the real part of the frequency is exactly intermediate between the limiting values for zero and infinite collision times. The absorption is approximately linear with λ at the low-pressure limit, varying as $\dfrac{p^2 a^2}{\omega_p^2} \dfrac{\lambda}{\omega_p}$, while at high pressures it is hyperbolic, varying as $\dfrac{p^2 a^2}{\omega_p^2} \dfrac{\omega_p}{\lambda}$. The negative frequency root of Eq. (67) yields

$$\omega = - \left[\omega_p + \frac{p^2 a^2}{2 \omega_p^2} \frac{1}{\lambda^2 + \omega_p^2} \left\{ \lambda^2 + 3 \omega_p^2 - i \ 2 \lambda \omega_p \right\} \right] \quad ,$$

and gives absorption and dispersion which corresponds completely to the results for the positive frequency case. It is important to note that the absolute magnitude of the damping is very small since it is proportional to $\dfrac{p^2 a^2}{\omega_p^2}$. For example, for $pa \simeq \omega_p/100$ the maximum damping frequency is less than $10^{-4} \omega_p$.

The dispersion relation Eq. (68) gives adequate information for $pa \ll \omega_p$. We are also interested in the case where pa is approximately or greater than ω_p, i.e., waves of length comparable to the Debye length. For this case the development in powers of pa/ω_p is not appropriate; one must also have detailed numerical information concerning $F(\Omega)$. In Figs. 3 and 4 we plot the real and imaginary parts of ω for the case $pa = \omega_p$ (waves equal to the Debye length) as a function of the collision frequency. The results are the same as for the long-wavelength case when $\lambda \gg \omega_p$. However, at values of λ smaller than ω_p the very large "drift" damping discussed in Sec. 4 manifests itself.
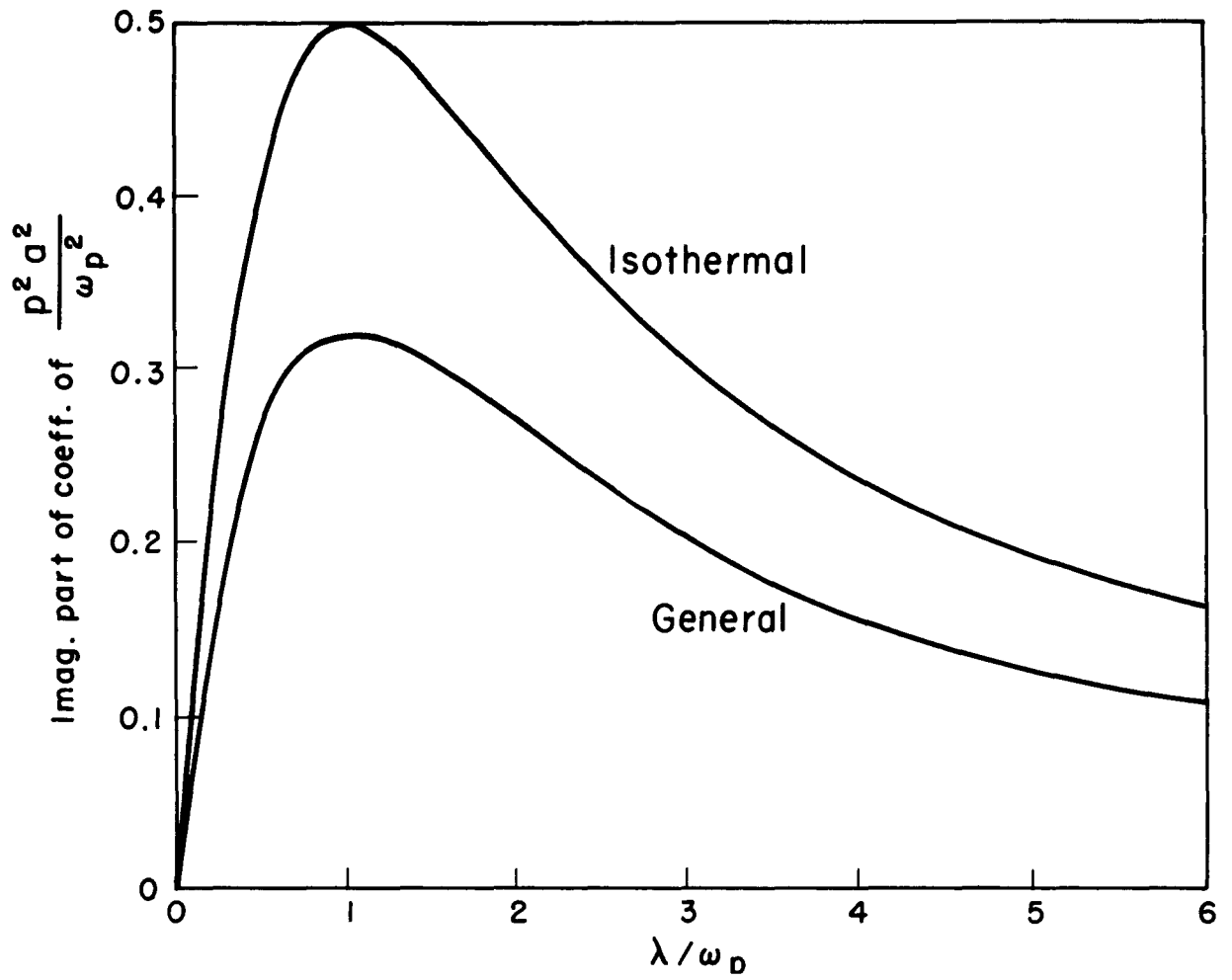
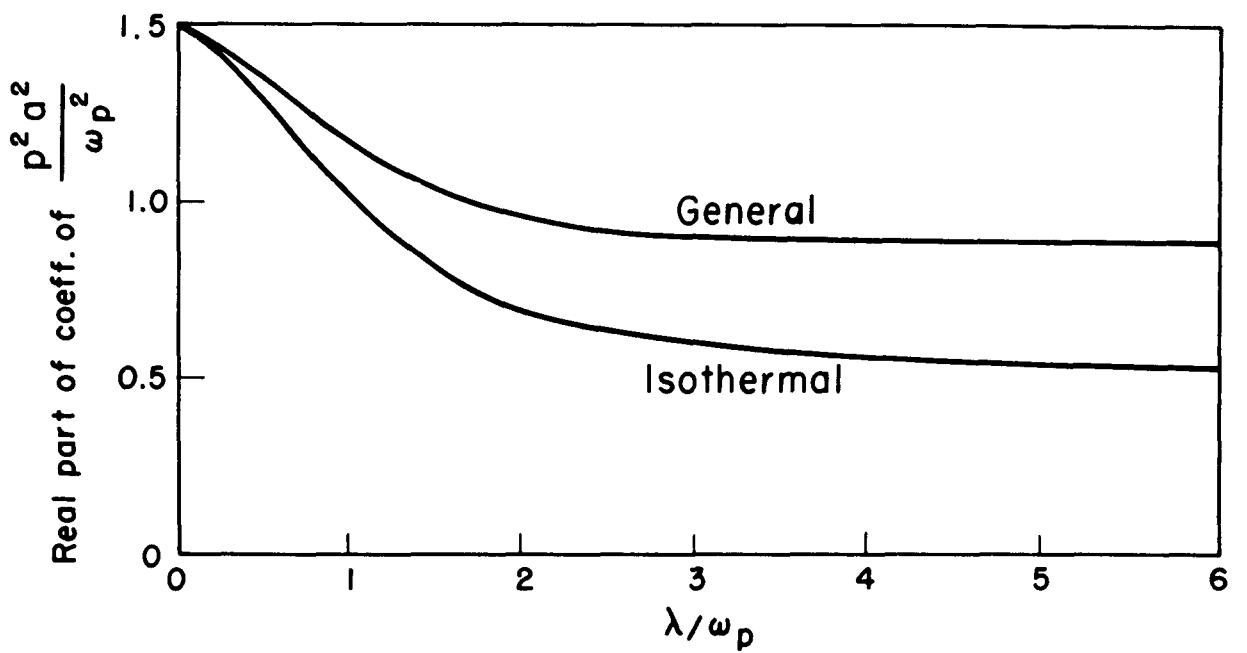Fig. 1. Absorption of ionized gas as function of collision frequency.

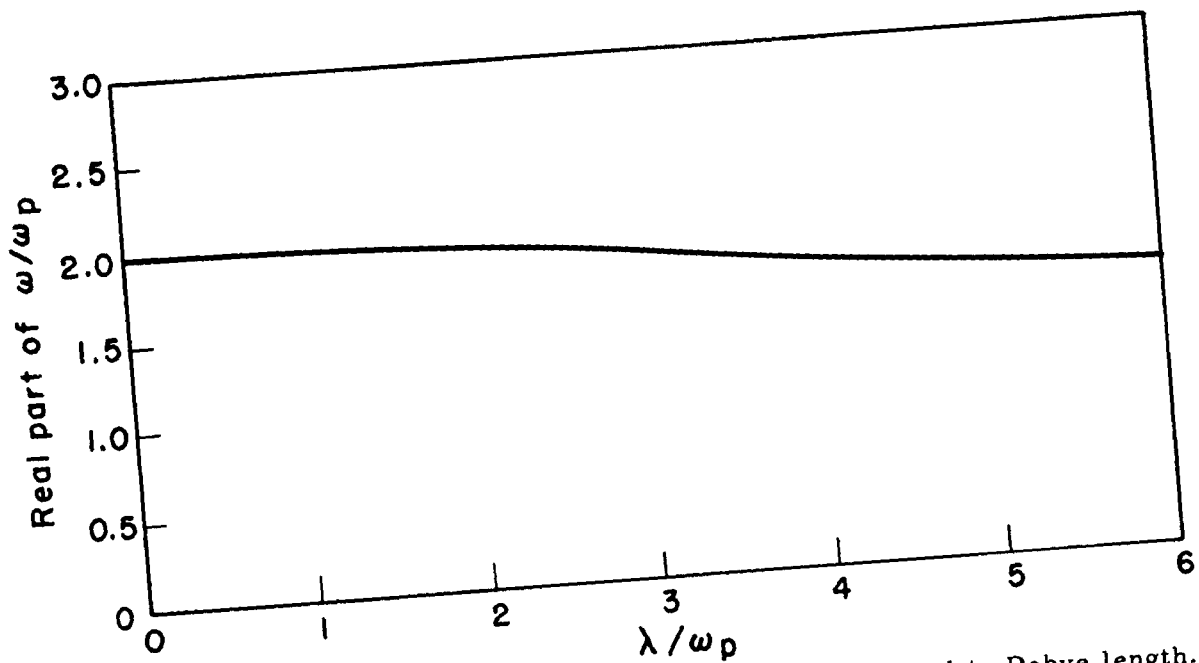Fig. 2. Dispersion of ionized gas as function of collision frequency.

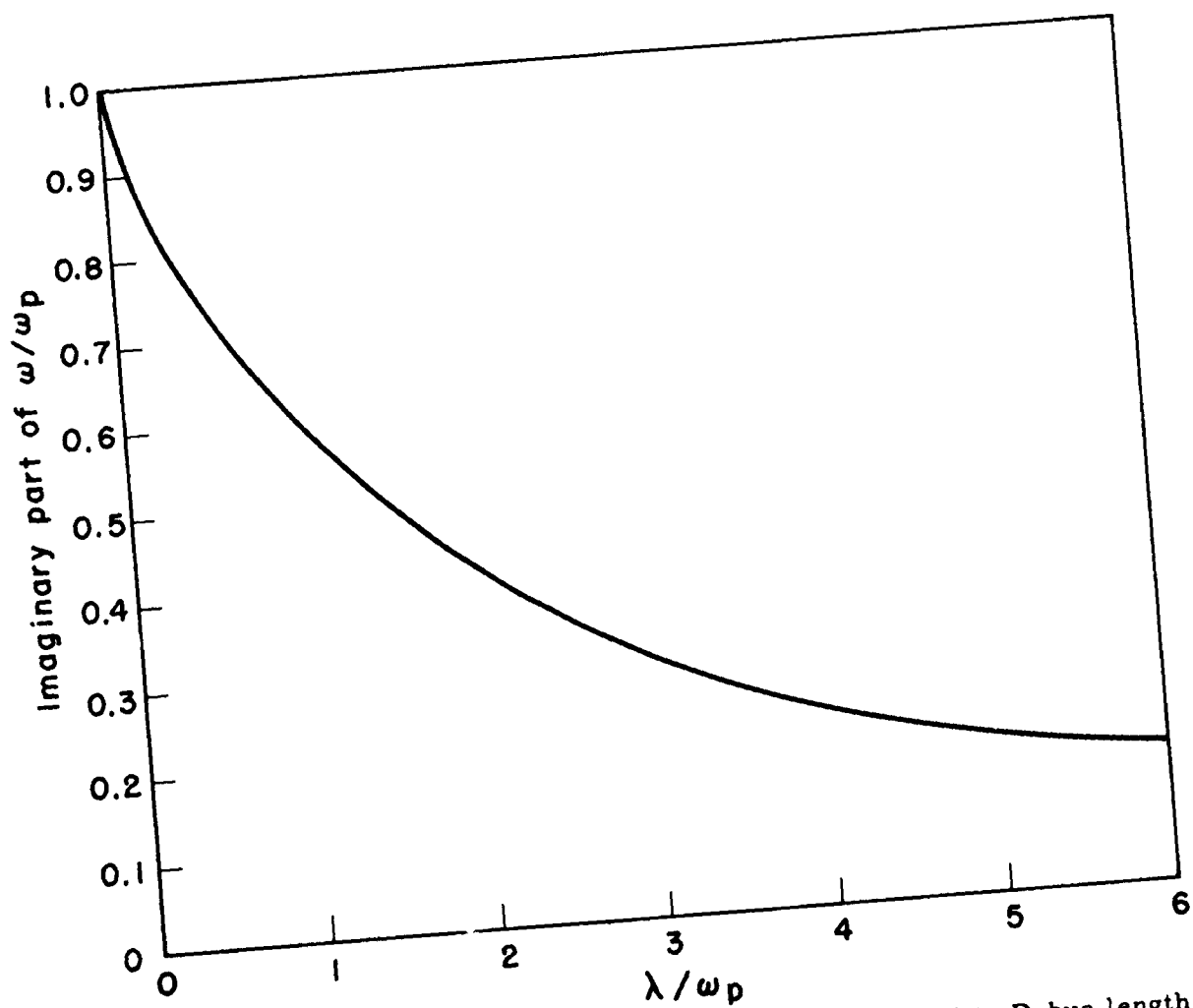Fig. 3. Dispersion of ionized gas for wavelength equal to Debye length.

Fig. 4. Absorption of ionized gas for wavelength equal to Debye length.

7. OSCILLATIONS OF AN UNCHARGED GAS

The general results of Secs. 3 to 5 apply to the initial-value problem for uncharged gases; the plasma frequency ω_p is then zero. For our constant-collision time, isothermal model the general dispersion relation Eq. (58) becomes, in ordinary units,

$$-\frac{p^2 a^2}{i\omega\lambda} = 1 + \left[\frac{pa}{\omega} + \frac{\omega + i\lambda}{pa}\right] i(2)^{1/2} F(\Omega) \qquad (69)$$

$$\Omega = \frac{\lambda - i\omega}{pa(2)^{1/2}}$$

As discussed in Sec. 5 the asymptotic expansion Eq. (61) is valid provided $pa \ll \lambda$, i.e., the frequency of the sound wave is smaller than the collision frequency. The first few terms of this expansion yield the result

$$\omega^2 + p^2 a^2 \left\{-\frac{1 + 4i\omega/\lambda}{(1 - i\omega/\lambda)^2}\right\} + \frac{p^4 a^4}{\lambda^2} \left\{\frac{3 - 18i\omega/\lambda}{(1 - i\omega/\lambda)^4}\right\} = 0 \qquad . \qquad (70)$$

In the limit of infinite collision frequency we find $\omega^2 = p^2 a^2$, the correct isothermal sound speed. The correction for finite, but large collision frequency, expressed in powers of pa/λ, is

$$\omega = pa \left\{1 - i\frac{pa}{\lambda} + \frac{1}{2}\left(\frac{pa}{\lambda}\right)^2\right\} \qquad . \qquad (71)$$

This expression indicates both an absorption and dispersion of sound for finite value of the collision frequency. The physical situation is similar to the Langmuir description of the damping of plasma waves with wavelength near to the Debye length.[8] Here, because of the translational motion, fast moving particles can occasionally move ahead of the sound wave from one region to a distant region without undergoing a collision and thus carry ordered motion from one region to another where the phase for ordered motion is different. The ordered energy is then dissipated into random thermal energy by collisions. This effect, of course, increases with increasing mean free path and with decreasing wavelength.

The series expansion Eq. (70) breaks down when pa $\simeq$ λ and one must resort to numerical evaluation of Eq. (69). The results for the real and imaginary parts are presented in Fig. 5. The accuracy of these results is, however, limited by the inadequate tables of the function $F(\Omega)$ available to us at present.

According to the discussion of Sec. 4 the asymptotic behavior of the distribution function differs according as $\sigma_o = -i\omega_o = -\lambda - ipu$, or $\sigma_1 = -i\omega_1$, as given by the dispersion relation Eq. (69), has the larger real part. The second case is what we have previously called the "collision-damping" case. For this case the damping frequency is less than the collision frequency and the asymtotic behavior of the distribution function is the same as that of the density. From Fig. 5 we see that this is the situation when pa $<$ 1.5λ. For the first case the damping frequency is greater than the collision frequency, the distribution function and density have different asymptotic behaviors, and the distribution function is directly sensitive to the initial value of the distribution function. The individual features of the particular initial value or boundary-value problem treated are then important.

For the "collision-damping" case, one may study the physical processes with the aid of the distribution function Eq. (49). Using the value of ω given by Eq. (71) we see that the deviation from the Maxwellian distribution is greatest for fast moving particles and short wavelengths. From Eq. (39) for the mass velocity, we see that the greatest contribution to the out-of-phase component of mass velocity comes from fast moving particles and short wavelengths in accordance with the picture discussed above.

8. GENERAL TREATMENT

In this section we shall remove the restriction made previously and allow for temperature fluctuations. The problem is now characterized by the full set of Eqs. (15) to (20). The passage to the linear approximation involves the
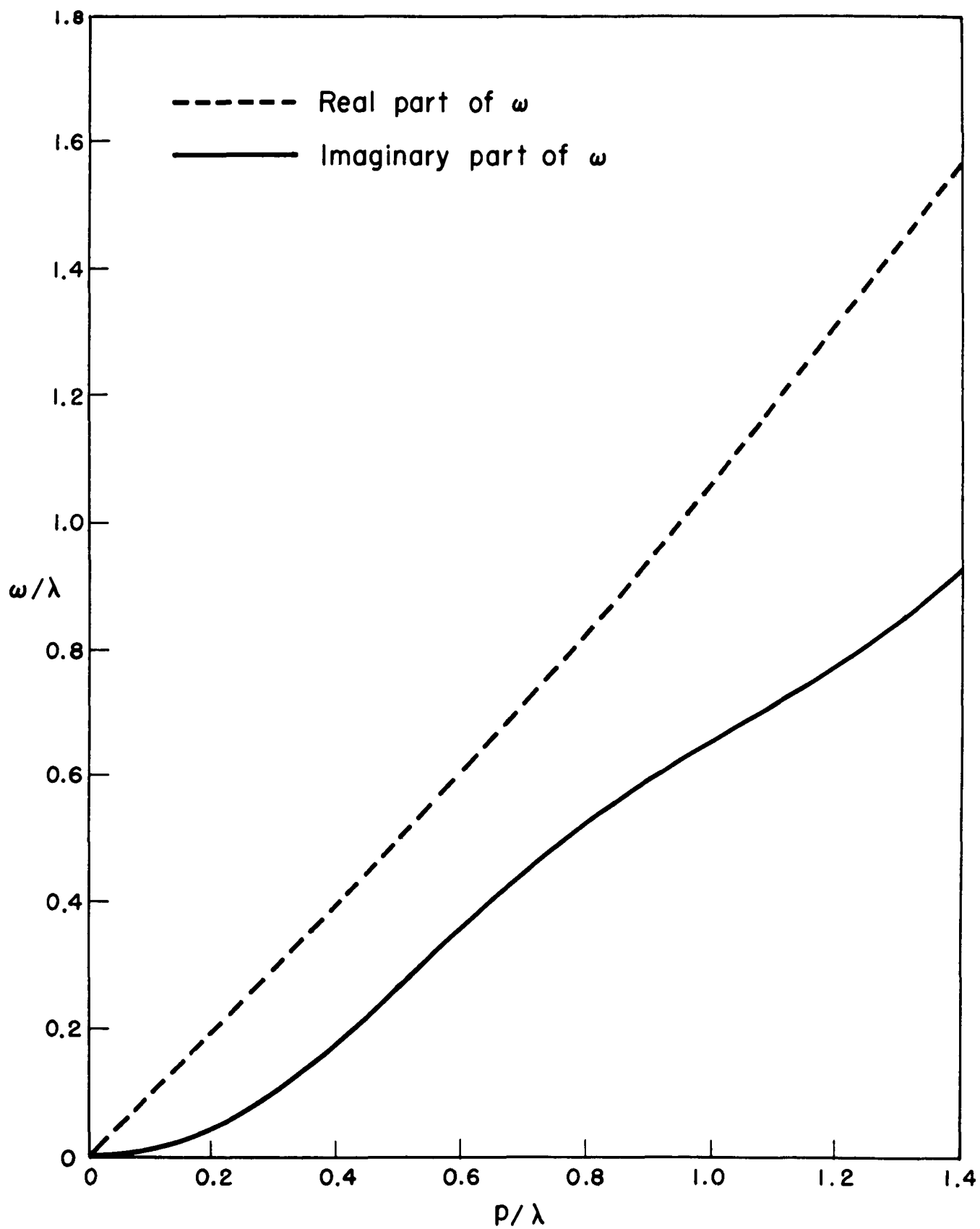
Fig. 5. Absorption and dispersion of sound for isothermal case.

additional fluctuating quantity $\tau(\vec{x}, t)$, defined by

$$T(\vec{x}, t) = T_0 \left\{ 1 + \tau(\vec{x}, t) \right\} \tag{72}$$

Equations (21) to (25) remain valid in the general case, but the expansion of $\boldsymbol{\Phi}$ in Eq. (26) must be replaced by

$$\boldsymbol{\Phi} = \bar{F} \left\{ 1 + \frac{m}{kT_0} \vec{v} \cdot \vec{q} + \tau \left(\frac{mv^2}{2kT_0} - \frac{3}{2} \right) \right\} \tag{73}$$

Equation (19) defining the temperature leads to

$$\frac{3kT_0}{m} (\nu + \tau) = \int v^2 \bar{F} \phi \, d\vec{v} \tag{74}$$

The basic set of linear equations governing the Fourier transformed variables for a disturbance in the x direction is

$$\left. \begin{array}{l} \dfrac{\partial \phi_{\vec{p}}}{\partial t} + ipu \, \phi_{\vec{p}} + \dfrac{e(E_{\vec{p}})_x}{kT_0} u = \lambda \left\{ \nu_{\vec{p}} - \phi_{\vec{p}} + \dfrac{m}{kT_0} u(q_{\vec{p}})_x + \tau_{\vec{p}} \left(\dfrac{mv^2}{2kT_0} - \dfrac{3}{2} \right) \right\} \\[2mm] \nu_{\vec{p}} = \int \phi_{\vec{p}} \bar{F} \, d\vec{v} \\[2mm] (q_{\vec{p}})_x = \int u \, \phi_{\vec{p}} \bar{F} \, d\vec{v} \\[2mm] \dfrac{3kT_0}{m} (\nu_{\vec{p}} + \tau_{\vec{p}}) = \int v^2 \phi_{\vec{p}} \bar{F} \, d\vec{v} \\[2mm] ip(E_{\vec{p}})_x = -4\pi en_0 \, \nu_{\vec{p}} \end{array} \right\} \tag{75}$$

One may eliminate $(E_{\vec{p}})_x$ and $(q_{\vec{p}})_x$ in the same way as in the isothermal case. Corresponding to Eq. (44) one now finds a coupled set of equations for the Laplace transforms of $\nu_{\vec{p}}$ and $\tau_{\vec{p}}$. This set is

$$\nu_{\vec{p}, \sigma} \left\{ 1 - \int \frac{\bar{F} \, d\vec{v} \left[\lambda - \frac{\lambda m}{kT_0} \frac{u\sigma}{ip} + \frac{k_0^2 u}{ip} \right]}{\lambda + \sigma + ipu} \right\} - \tau_{\vec{p}, \sigma} \left\{ \int \frac{\bar{F} \, d\vec{v} \left[\frac{mv^2}{2kT_0} - \frac{3}{2} \right] \lambda}{\lambda + \sigma + ipu} \right\}$$

$$= \int \frac{\bar{F} \, d\vec{v} \left(\frac{\lambda m}{kT_0} \frac{u \, \nu_{\vec{p}}(0)}{ip} + \phi_{\vec{p}}(0) \right)}{\lambda + \sigma + ipu} \tag{76}$$

$$\nu_{\vec{p},\sigma}\left\{1 - \frac{m}{3kT_o}\int\frac{v^2\,\bar{F}\,d\vec{v}}{\lambda + \sigma + ipu}\left[\lambda - \frac{\lambda m}{kT_o}\frac{u\sigma}{ip} + \frac{k_o^2 u}{ip}\right]\right\}$$

$$+ \tau_{\vec{p},\sigma}\left\{1 - \frac{\lambda m}{3kT_o}\int\frac{v^2\,\bar{F}\,d\vec{v}}{\lambda + \sigma + ipu}\left(\frac{mv^2}{2kT_o} - \frac{3}{2}\right)\right\} \tag{77}$$

$$= \frac{m}{3kT_o}\int\frac{v^2\,\bar{F}\,d\vec{v}}{\lambda + \sigma + ipu}\left\{\frac{\lambda m}{kT_o}\frac{u\,\nu_{\vec{p}}(0)}{ip} + \phi_{\vec{p}}(0)\right\}\qquad .$$

The dispersion relation in dimensionless units is then

$$\left\{1 - \frac{1}{(2\pi)^{3/2}}\iiint\frac{e^{-v^2/2}\,d\vec{v}}{\lambda + i(pu - \omega)}\left[\lambda + \frac{\lambda u\omega}{p} + \frac{u}{ip}\right]\right\}$$

$$\left\{1 + \frac{\lambda}{6(2\pi)^{3/2}}\iiint\frac{e^{-v^2/2}\,v^2(3 - v^2)}{\lambda + i(pu - \omega)}\right\} = \left\{1 - \frac{1}{3(2\pi)^{3/2}}\right. \tag{78}$$

$$\left.\iiint\frac{e^{-v^2/2}\,v^2\,d\vec{v}}{\lambda + i(pu - \omega)}\left(\lambda + \frac{\lambda u\omega}{p} + \frac{u}{ip}\right)\right\}\left\{\frac{\lambda/2}{(2\pi)^{3/2}}\iiint\frac{e^{-v^2/2}(3 - v^2)\,d\vec{v}}{\lambda + i(pu - \omega)}\right\}.$$

In order to study this dispersion relation we express all the integrals

occuring in terms of the single integral

$$\frac{1}{(2\pi)^{1/2}}\int_c\frac{e^{-\zeta^2/2}\,d\zeta}{\zeta - \theta} = i\sqrt{2}\,F(\Omega)$$

which was discussed in Sec. 5.

We find

$$\frac{1}{(2\pi)^{3/2}}\iiint_c\frac{e^{-v^2/2}\,d\vec{v}}{\lambda + i(pu - \omega)} = \frac{\sqrt{2}}{p}\,F(\Omega)$$

$$\frac{1}{(2\pi)^{3/2}}\iiint\frac{e^{-v^2/2}\,v^2\,d\vec{v}}{\lambda + i(pu - \omega)} = \frac{\sqrt{2}}{p}\left\{\Omega + 2(1 - \Omega^2)\,F(\Omega)\right\}$$

$$\frac{1}{(2\pi)^{3/2}}\iiint\frac{e^{-v^2/2}\,v^4\,d\vec{v}}{\lambda + i(pu - \omega)} = \frac{\sqrt{2}}{p}\left\{5\Omega - 2\Omega^3 + 4(\Omega^4 - 2\Omega^2 + 2)\,F(\Omega)\right\} \tag{79}$$

$$\frac{1}{(2\pi)^{3/2}}\iiint\frac{e^{-v^2/2}\,u\,v^2\,d\vec{v}}{\lambda + i(pu - \omega)} = \frac{1}{ip}\left\{3 - 2\Omega^2 - 4\Omega(1 - \Omega^2)\,F(\Omega)\right\}$$

$$\frac{1}{(2\pi)^{3/2}}\iiint\frac{e^{-v^2/2}(3 - v^2)\,d\vec{v}}{\lambda + i(pu - \omega)} = \frac{\sqrt{2}}{p}\left\{-\Omega + (1 + 2\Omega^2)\,F\right\}$$

$$\frac{1}{(2\pi)^{3/2}} \iiint \frac{e^{-v^2/2}(3-v^2)v^2\,d\vec{v}}{\lambda + i(pu-\omega)} = \frac{\sqrt{2}}{p}\left\{-2\Omega + 2\Omega^3 + (-2 + 2\Omega^2 - 4\Omega^4)\,F\right\}\Big].$$

We also find

$$\frac{1}{(2\pi)^{3/2}} \iiint \frac{e^{-v^2/2}\,d\vec{v}}{\lambda + i(pu-\omega)}\left[\lambda + \frac{\lambda u\omega}{p} + \frac{u}{ip}\right] = -\frac{(1+\lambda^2 - p\lambda\sqrt{2}\,\Omega)}{p^2}$$

$$+ F(\Omega)\left\{\frac{\lambda\sqrt{2}}{p} + \frac{2\Omega}{p^2}(1 + \lambda^2 - p\lambda\sqrt{2}\,\Omega)\right\} \quad , \tag{80}$$

$$\frac{1}{(2\pi)^{3/2}} \iiint \frac{e^{-v^2/2}v^2\,d\vec{v}}{\lambda + i(pu-\omega)}\left[\lambda + \frac{\lambda u\omega}{p} + \frac{u}{ip}\right] = \frac{\lambda\sqrt{2}}{p}\left\{\Omega + 2(1-\Omega^2)\,F(\Omega)\right\}$$

$$- \frac{1}{p^2}(1 + \lambda^2 - p\lambda\sqrt{2}\,\Omega)\left[3 - 2\Omega^2 - 4\Omega(1-\Omega^2)\,F\right].$$

Using these results we can write the dispersion relation as

$$\left\{1 + \frac{(1+\lambda^2 - p\lambda\sqrt{2}\,\Omega)}{p^2} - F(\Omega)\left[\frac{\lambda\sqrt{2}}{p} + \frac{2\Omega}{p^2}(1 + \lambda^2 - p\lambda\sqrt{2}\,\Omega)\right]\right\}$$

$$\left\{1 + \frac{\lambda}{6}\frac{\sqrt{2}}{p}\left[-2\Omega + 2\Omega^3 + (-2 + 2\Omega^2 - 4\Omega^4)\,F\right]\right\} \tag{81}$$

$$= \left\{1 - \frac{1}{3}\frac{\lambda\sqrt{2}}{p}\left[\Omega + 2(1-\Omega^2)\,F\right] + \frac{1}{3p^2}(1+\lambda^2 - p\lambda\sqrt{2}\,\Omega)\left[3 - 2\Omega^2 - 4\Omega(1-\Omega^2)F\right]\right\}$$

$$\left\{\frac{\lambda}{2}\frac{\sqrt{2}}{p}\left[-\Omega + (1 + 2\Omega^2)\,F\right]\right\} \quad .$$

The dispersion relation Eq. (81) for the general case is considerably more involved than the relation Eq. (58), which holds in the isothermal case. However, as emphasized in Sec. 5 one can obtain simple results for wavelengths long compared to either the Debye length or the mean free path. For this case the asymptotic development for $F(\Omega)$ holds and one finds (in dimensionless units)

$$(1-\omega^2)(-i\omega) + \frac{\lambda p^2}{(\lambda - i\omega)^2}\left\{\frac{5}{3} - \frac{19\omega^2}{3} - \frac{5i\omega\lambda}{3} + \frac{3i\omega}{\lambda}\right\}$$

$$\tag{82}$$

$$+ \frac{p^4}{(\lambda - i\omega)^4}\left\{-5\lambda^3 - \frac{20}{3}i\omega\lambda^2 + \lambda(7 + 3\omega^2) - 20(1-\omega^2) - 15i\omega\right\} = 0 \quad .$$

For the study of the oscillations of an ionized gas we obtain an expression for the frequency which corresponds to Eq. (66). Keeping terms of order $p^2 a^2 / \omega_p^2$ we find for the frequency

$$\omega = \omega_p + \frac{p^2 a^2}{2\omega_p} \frac{1}{(\lambda^2 + \omega_p^2)} \left\{ \left(\frac{5}{3} \lambda^2 + 3\omega_p^2 \right) - i \frac{4}{3} \lambda \omega_p \right\} \quad . \quad (83)$$

At high pressures the collision frequency λ is large compared to the plasma frequency and one finds $\omega \to \omega_p + \frac{p^2 a^2}{2\omega_p} \cdot \frac{5}{3}$. This is the value found previously by a generalization of the Thomson method[11] and verifies the high-pressure nature of the transport type of procedure. At low pressures we find $\omega \to \omega_p + \frac{p^2 a^2}{2\omega_p^2} \cdot 3$, the correct dispersion relation for waves of length large compared to the Debye length. The maximum damping frequency for the general case occurs again when the collision frequency is equal to the plasma frequency. It is in magnitude 2/3 that for the isothermal case; the decreased absorption accompanies the smaller dispersion for the general case. The detailed variation of frequency as a function of collision frequency as given by Eq. (83) is plotted in Figs. 1 and 2. In the event that the wavelength is comparable to both the collision frequency and the Debye length one must resort to numerical information concerning $F(\Omega)$. Tables of sufficient accuracy to solve the complicated Eq. (81) are not available to us at present; we leave the detailed investigation to a later time.

For an uncharged gas the expression Eq. (82) reduces to

$$\omega^2 + \frac{p^2 a^2}{(1 - i\omega/\lambda)^2} \left\{ \frac{19}{3} \frac{i\omega}{\lambda} - \frac{5}{3} \right\} + \frac{p^4 a^4}{\lambda^2 (1 - i\omega/\lambda)^4} \left\{ \frac{5i\lambda}{\omega} - \frac{20}{3} - \frac{3i\omega}{\lambda} \right\} = 0. \quad (84)$$

Developing in powers of pa/λ, one finds for $pa/\lambda \ll 1$

$$\omega = \left(\frac{5}{3} \right)^{1/2} pa \left\{ 1 - i \left(\frac{5}{3} \right)^{1/2} \frac{pa}{\lambda} \cdot \frac{9}{5} + 11.3 \frac{p^2 a^2}{\lambda^2} \right\} \quad . \quad (85)$$

Thus one finds both an absorption and a dispersion of sound waves. No attempt has been made in this paper to connect our collision frequency λ with molecular parameters; it is known therefore only to within a factor of the order of unity.

This comparison will be undertaken in Paper 4, where we also discuss models more realistic than the present constant-collision-time model. For the present we note that if one fits λ by taking the absorption equal to that predicted by the Navier-Stokes equations (and also the Burnett equations) one finds $\lambda = \frac{90}{35} \frac{a^2}{\eta}$, where $\eta = \frac{1}{2} \rho_o \left(\frac{8kT}{\pi m}\right)^{1/2} \ell$; η is the coefficient of viscosity; ℓ is the mean free path. The coefficient of the dispersive term in Eq. (85) is then 25 percent higher than that given by the Navier-Stokes equation; it thus lies between the predictions of the Navier-Stokes and of the Burnett equations. The interesting experimental observations of Greenspan on the dispersion and absorption of helium gas reveal important discrepancies between experiment and the predictions of both the Navier-Stokes and the Burnett equations when pa $\simeq \lambda$. From the point of view of the present work the study of this region requires a numerical analysis of Eq. (81). We shall therefore later take up the study of the translational dispersion in a more detailed way.